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Three-Dimensional Relativistic-Covariant Poisson Brackets

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Using the algebraic properties of Poisson brackets, we extend the three-dimensional brackets (for a single free particle) to conform to the demands of special relativity. This yields, in an essentially unique way, the manifestly covariant extension $[x_\mu, p_\nu] = \delta_{\mu\nu} + p_\mu p_\nu/m^2c^2$. Position and time then become fully dynamical variables expressible in terms of the canonical conjugate q_i and p_i and the time parameter θ as $x_i = q_i + p_i(\mathbf{q}\cdot\mathbf{p})/m^2c^2$ and $t = \theta + E(\mathbf{q}\cdot\mathbf{p})/m^2c^4$. In the quantized version, the length associated with a particle of mass m is shown to be an integral multiple of the Compton wavelength $\lambda_C = \hbar/mc$.

1. INTRODUCTION AND SUMMARY

Traditionally, the algorithm for quantization of a classical system proceeds by first casting the dynamics into the Hamiltonian Poisson brackets (PB) form (Dirac, 1958). The classical dynamical variables are then replaced by Hermitian operators satisfying commutation relations such that the quantum commutator of a pair corresponds to the classical PB of the pair. As is well known, this procedure is covariant under Galilean transformations from one inertial frame to another. In contradistinction, the PB (or the corresponding commutators) are changed by a Lorentz boost. Thus, taking the simplest case of a single free particle with energy $E = (\mathbf{p}^2 + m^2)^{1/2}$, we find that under the transformation

$$\begin{aligned} x' &= \gamma(x - ut), \qquad t' = \gamma(t - ux) \\ p'_x &= \gamma(p_x - uE) \qquad E' = \gamma(E - up_x) \\ \gamma &= (1 - u^2)^{-1/2} \end{aligned} \tag{1}$$

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the PB $[x, p_x] = 1$ transforms into

$$[x', p'_x] = [\gamma(x - ut), \gamma(p_x - uE)] = \gamma^2(1 - up_x/E) = \gamma E'/E \qquad (2)$$

which differs from 1 if the velocity u is not zero. (We are using units such that $c = \hbar = 1$. Occasionally we will restore c and $\hbar$ in the formulas.) It is thus apparent that, even in the simplest case, *the Einsteinian principle of relativity is incompatible with the Hamiltonian formalism.* {Nor is it difficult to see the general reason for this incompatibility. Under canonical transformations, which are special transformations of phase space *at a given time*, the PB transform homomorphically, that is, if $A(t) \to A'(t)$ and $B(t) \to B'(t)$, then $[A(t), B(t)] \to [A'(t), B'(t)]$. But in order for special relativity to prevail we need this correspondence *at different times*, namely, $[A(t), B(t)] \to [A'(t'), B'(t')]$.}

Facing this impasse, the general attitude was to endow the time parameter the status of an independent fourth dynamical coordinate on the same footing as that of the space coordinates. Some of the difficulties encountered in this approach are detailed in Goldstein (1980). Our goal in the present work is less ambitious. Rather than imposing a totally new role on the time as a fourth dynamical coordinate, we shall stay within the familiar three-dimensional world [Minkowski notwithstanding[2]], and ask whether we can use the algebraic properties of the PB to extend the brackets to higher velocities in a way compatible with special relativity. The requirement of covariance will then be used to *derive* rather than to *impose* conditions on the time and space variables. It should be noted that there is nothing in the Lorentz transformation or in Minkowski's 4-world formulation which entails an identical role to time and space. After all, time enters as a fourth component of a world vector with an imaginary coefficient or with a metric different from that of space.

It turns out that the sought three-dimensional extension of PB can be achieved in an (essentially) unique way yielding the manifestly covariant extension

$$[x_\mu, p_\nu] = \delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2 c^2} \qquad (3)$$

We use 4-vectors such as $(x_\mu) = (\mathbf{x}, ict)$, $(p_\mu) = (\mathbf{p}, iE/c)$. Roman indices $i, j, \ldots$ denote space components 1, 2, 3. Greek indices $\mu, \nu, \ldots$ run through 1, 2, 3, 4. As equation (3) shows, the first corrections due to higher velocities are of the order of v^2/c^2. An amusing (in our language "superrelativistic") treatment of equation (3) retaining only $p_x p_x/m^2c^2$ on the r.h.s.

[2]"Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality" (Minkowski, 1908).

appears in a different context ("quantization in the large") in the work of Greenberger (1983). As can be checked using Jacobi's identity, validity of equation (3) with $[p_i, p_j] = 0$ implies $[x_i, x_j] \neq 0$ for $i \neq j$. Thus in our relativistic quantum mechanics the x_i no longer commute and a position representation is not possible. In fact, in terms of the canonical coordinates q_i and p_i, where q_i is the coordinate conjugate to the momentum p_i satisfying $[q_i, p_j] = \delta_{ij}$ and $[q_i, q_j] = 0$, the space and time variables become (like the angular momentum $\mathbf{L} = \mathbf{q} \times \mathbf{p}$) dynamical variables expressible as $x_i = x_i(q, p)$, $t = t(q, p)$. Classically, we find

$$x_i = q_i + \frac{p_i}{m^2c^2}(\mathbf{q} \cdot \mathbf{p}) \tag{4}$$

and

$$t = \theta + \frac{E}{m^2c^4}(\mathbf{q} \cdot \mathbf{p}) \tag{5}$$

where the time parameter θ can be chosen to be a "c-number" independent of q and p. It can be shown that under the Lorentz transformation (1), the product θE remains invariant, that is,

$$\theta E = \theta' E' = \tau mc^2 \tag{6}$$

where τ is the proper time of the particle. In quantum mechanics x_i and t are supposed to be Hermitian operators. This can be achieved by symmetrization of equations (4) and (5). Thus, for example,

$$x_i = q_i + \frac{1}{2m^2c^2}\{p_i(\mathbf{q} \cdot \mathbf{p}) + (\mathbf{p} \cdot \mathbf{q})p_i\} \tag{4'}$$

In terms of the time parameter θ, the equation of motion of a dynamical variable $A(q, p, \theta)$ is

$$i\hbar \frac{d}{d\theta} A(q, p, \theta) = i\hbar \frac{\partial A}{\partial \theta} + [A, E] \tag{7}$$

For example,

$$i\hbar \frac{dt}{d\theta} = i\hbar \frac{\partial t}{\partial \theta} + [t, E] = i\hbar\left(1 + \frac{v^2/c^2}{1 - v^2/c^2}\right) = i\hbar \frac{1}{1 - v^2/c^2} \tag{8}$$

Since position and time no longer commute, it is meaningless to ask about the amplitude to be found at $\mathbf{x}$ at time t. Rather, one could legitimately ask about the amplitude $\psi(\mathbf{q}, \theta)\, d^3q$ [or $\psi(\mathbf{p}, \theta)\, d^3p$] to find the particle at time θ within d^3q around $\mathbf{q}$ (d^3p around $\mathbf{p}$). We end our summary of partial results by the following surprise. The fact that $\mathbf{x}$ behaves like a vector under rotations together with the commutation relation $[x, y] = (i\hbar/m^2c^2)L_z$, etc., entail quantization of the length. Thus, for a particle of mass m, the length is measured in units of Compton wavelength $\lambda_C = \hbar/mc$.

Of course, all this is very preliminary, pertaining to a single free particle. Where it could lead to is not at all clear. Nevertheless, I find the partial results intriguing enough to be disclosed to the public even at this stage. In the next sections I bring the detailed arguments leading to the results quoted above.

2. DERIVATION OF THE EXTENDED POISSON BRACKETS (3)

In this section we use the algebraic properties of PB, namely

$$\begin{aligned} [A, B] &= -[B, A] \\ [A, \beta B + \gamma C] &= \beta[A, B] + \gamma[A, C] \\ [A, BC] &= [A, B]C + B[A, C] \\ [[A, B], C] + [[C, A], B] &+ [[B, C], A] = 0 \end{aligned} \tag{9}$$

together with the requirements of covariance to extend the PB to higher velocities. Our derivation assumes classical commuting quantities, but the same derivation applies *mutatis mutandis* to operators (working in the $\mathbf{v}$ representation). It is natural to try first the following assumption: $[p_i, p_j] = 0$, $[x, p_y] = [x, p_z] = 0$, and $[x, p_x] = f(\mathbf{p})$, where $f(\mathbf{p}) \to 1$ as $\mathbf{p} \to \mathbf{0}$ to regain the low-velocity limit. It fails. Indeed, with the aid of equations (9) we find that the energy $E = (\mathbf{p}^2 + m^2)^{1/2}$ (expanded in powers of the p_i) satisfies

$$[x, E] = f(\mathbf{p}) \frac{\partial E}{\partial p_x} = f(\mathbf{p}) \frac{p_x}{E} \tag{10}$$

Also, from $[x', p'_y] = [\gamma(x - ut), p_y] = 0$ it follows that $[t, p_y] = 0$. Hence,

$$\begin{aligned} [t', p'_x] &= [\gamma(t - ux), \gamma(p_x - uE)] \\ &= -\gamma^2 u f(\mathbf{p})(1 - u p_x/E) = -u\gamma f(\mathbf{p}) E'/E \neq 0 \end{aligned}$$

unless $u = 0$.

Our next try is to work with velocities rather than momenta. That is, we assume $[v_i, v_j] = 0$, $[x, v_y] = [x, v_z] = 0$, and $[x, v_x] = f(\mathbf{v})/m$, where $f(0, 0, 0) = 1$. From this assumption and equations (9) it follows that for any analytic function $g(\mathbf{v}) \equiv g(v_x, v_y, v_z)$ we must have

$$[x, g(\mathbf{v})] = \frac{1}{m} f(\mathbf{v}) \frac{\partial g}{\partial v_x} \tag{11}$$

In particular, since

$$\frac{\partial}{\partial v_x}\frac{1}{(1-v^2)^{1/2}}=\frac{v_x}{(1-v^2)^{3/2}}$$

we find for the components of momentum $\mathbf{p}$ and the energy E, where

$$p_i=\frac{mv_i}{(1-v^2)^{1/2}} \qquad \text{and} \qquad E=\frac{m}{(1-v^2)^{1/2}}=(\mathbf{p}^2+m^2)^{1/2} \tag{12}$$

that

$$\begin{aligned}
[x,p_x]&=f(\mathbf{v})\left[\frac{1}{(1-v^2)^{1/2}}+\frac{v_x^2}{(1-v^2)^{3/2}}\right]=f(\mathbf{v})\frac{1-v_y^2-v_z^2}{(1-v^2)^{3/2}}\\
[x,p_y]&=f(\mathbf{v})\frac{v_xv_y}{(1-v^2)^{3/2}}\\
[x,p_z]&=f(\mathbf{v})\frac{v_xv_z}{(1-v^2)^{3/2}}\\
[x,E]&=f(\mathbf{v})\frac{v_x}{(1-v^2)^{3/2}}
\end{aligned} \tag{13}$$

We now demand covariance. Starting with

$$\begin{aligned}
[x',p_y']&=[\gamma(x-ut),p_y]=\gamma\left\{f(\mathbf{v})\frac{v_xv_y}{(1-v^2)^{3/2}}-u[t,p_y]\right\}\\
&=f(\mathbf{v}')\frac{v_x'v_y'}{(1-v'^2)^{3/2}}
\end{aligned} \tag{14}$$

and using the transformation properties of velocity under a boost in the x direction

$$\begin{aligned}
&v_x'=\frac{v_x-u}{1-uv_x}, \qquad v_y'=\frac{1}{\gamma}\frac{v_y}{1-uv_x}, \qquad v_z'=\frac{1}{\gamma}\frac{v_z}{1-uv_x}\\
&1-v'^2=\frac{(1-v^2)(1-u^2)}{(1-uv_x)^2}, \qquad \frac{1}{(1-v'^2)^{1/2}}=\frac{\gamma(1-uv_x)}{(1-v^2)^{1/2}}
\end{aligned} \tag{15}$$

we find, for the special case $u=v_x$ [which causes the r.h.s. of (14) to vanish], that

$$[t,p_y]=f(\mathbf{v})\frac{v_y}{(1-v^2)^{3/2}} \tag{16}$$

Substituting this result in equation (14) and using (15), we have

$$f(\mathbf{v})=f(\mathbf{v}')\gamma(1-uv_x) \qquad \text{for any} \quad u \tag{17}$$

Hence, for $u = v_x$ we obtain the functional equation

$$f(v_x, v_y, v_z) = (1 - v_x^2)^{1/2} f\left(0, \frac{v_y}{(1 - v_x^2)^{1/2}}, \frac{v_z}{(1 - v_x^2)^{1/2}}\right) \tag{18}$$

Similarly, demanding the covariance of $[x', p_z']$, we obtain

$$[t, p_z] = f(\mathbf{v}) \frac{v_z}{(1 - v^2)^{3/2}} \tag{19}$$

By symmetry one expects that

$$[t, p_x] = f(\mathbf{v}) \frac{v_x}{(1 - v^2)^{3/2}} \tag{20}$$

and

$$[t, E^2] = [t, p_x^2 + p_y^2 + p_x^2 + m^2] = 2f(\mathbf{v}) \frac{\mathbf{p} \cdot \mathbf{v}}{(1 - v^2)^{3/2}}$$

$$= 2f(\mathbf{v}) E \frac{v^2}{(1 - v^2)^{3/2}} = 2E[t, E] \tag{21}$$

and hence

$$[t, E] = f(\mathbf{v}) \frac{v^2}{(1 - v^2)^{3/2}} \tag{22}$$

It can be checked that these results indeed follow from the covariance of $[x', p_x']$ and $[x', E']$. Equations (16), (19), (20), and (22) show that $f(\mathbf{v}) \equiv f(v_x, v_y, v_z)$ must be a symmetric function of its arguments. We shall now show that the only solution of equation (18) which is a symmetric function of its arguments and satisfies $f(0, 0, 0) = 1$ is

$$f(\mathbf{v}) = (1 - v^2)^{1/2} \tag{23}$$

Indeed, exchanging $v_x \leftrightarrow v_y$ in equation (18), we find

$$f(v_x, v_y, v_z) = (1 - v_y^2)^{1/2} f\left(0, \frac{v_x}{(1 - v_y^2)^{1/2}}, \frac{v_z}{(1 - v_y^2)^{1/2}}\right) \tag{24}$$

Hence, for $v_x = 0$,

$$f(0, v_y, v_z) = (1 - v_y^2)^{1/2} f\left(0, 0, \frac{v_z}{(1 - v_y^2)^{1/2}}\right) \tag{25}$$

Reducing (25) further to a single independent argument by putting $v_z = 0$ (and exchanging $v_y \leftrightarrow v_z$), we finally have

$$f(0, v_y, 0) = f(0, 0, v_y) = (1 - v_y^2)^{1/2} f(0, 0, 0) = (1 - v_y^2)^{1/2} \tag{26}$$

Using equations (18), (25), and (26), we can successively reduce the number of independent arguments in $f(v_x, v_y, v_z)$ to obtain

$$\begin{aligned} f(v_x, v_y, v_z) &= (1-v_x^2)^{1/2} f\left(0, \frac{v_y}{(1-v_x^2)^{1/2}}, \frac{v_z}{(1-v_x^2)^{1/2}}\right) \\ &= (1-v_x^2)^{1/2}\left(1-\frac{v_y^2}{1-v_x^2}\right)^{1/2} \\ &\quad \times f\left(0, 0, \frac{v_z}{(1-v_x^2)^{1/2}} \frac{1}{(1-v_y^2/(1-v_x^2))^{1/2}}\right) \\ &= (1-v_x^2-v_y^2)^{1/2} f\left(0, 0, \frac{v_z}{(1-v_x^2-v_y^2)^{1/2}}\right) \\ &= (1-v_x^2-v_y^2)^{1/2}\left(1-\frac{v_z^2}{1-v_x^2-v_y^2}\right)^{1/2} \\ &= (1-v_x^2-v_y^2-v_z^2)^{1/2} = (1-v^2)^{1/2} \end{aligned} \tag{27}$$

Equations (13), (16), (19), (20), (22), and (23) determine completely the extended PB in terms of the velocities. Returning to the momenta via equation (12) and restoring the velocity of light c, we find the manifestly covariant expression

$$[x_\mu, p_\nu] = \delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2 c^2} \tag{28}$$

To obtain the corresponding quantum mechanical commutator, we must multiply the r.h.s. by $i\hbar$. Thus (using the same notation for PB and commutators),

$$[x_\mu, p_\nu] = i\hbar\left(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2 c^2}\right) \tag{29}$$

3. POSITION AND TIME AS DYNAMICAL VARIABLES

Equation (29) has been derived under the assumption $[p_i, p_j] = 0$ that the components of linear momentum commute. This can no longer be true for the components of the position vector x_i or, for that matter, the position and the time t. Indeed, using the Jacobi identity

$$[x_\sigma, [x_\mu, p_\nu]] + [p_\nu, [x_\sigma, x_\mu]] + [x_\mu, [p_\nu, x_\sigma]] = 0 \tag{30}$$

and equation (29), we find

$$[[x_\sigma, x_\mu], p_\nu] = -\frac{\hbar^2}{m^2 c^2}(\delta_{\sigma\nu} p_\mu - \delta_{\mu\nu} p_\sigma) \tag{31}$$

This result can be understood, provided we give up the identification of the position coordinate x_i and the time coordinate t with the canonical coordinate q_i and the time parameter θ. Indeed, returning to classical physics, let q_i denote the coordinate conjugate to p_i satisfying

$$[q_i, q_j] = 0, \qquad [q_i, p_j] = \delta_{ij} \tag{32}$$

and let us look for functions $x_i = x_i(q, p)$ and $t = t(q, p)$ such that equation (28) is satisfied. Using the original definition of PB, namely

$$[A, B] = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right) \tag{33}$$

we find

$$\begin{aligned} [x, p_x] &= \frac{\partial x}{\partial q_x} = 1 + \frac{p_x^2}{m^2c^2} \\ [x, p_y] &= \frac{\partial x}{\partial q_y} = \frac{p_x p_y}{m^2c^2} \\ [x, p_z] &= \frac{\partial x}{\partial q_z} = \frac{p_x p_z}{m^2c^2} \end{aligned} \tag{34}$$

The solution of these equations subject to the boundary condition $x \to q_x$ as $v/c \to 0$ is

$$x = q_x + \frac{p_x}{m^2c^2} \mathbf{q} \cdot \mathbf{p} \tag{35}$$

or, in general,

$$x_i = q_i + \frac{p_i}{m^2c^2} \mathbf{q} \cdot \mathbf{p} \tag{36}$$

Equation (36) can be easily inverted to express q_i in terms of x_j and p_j. Indeed, multiplying (36) by p_i and summing over i, we find

$$\mathbf{x} \cdot \mathbf{p} = \mathbf{q} \cdot \mathbf{p} \left(1 + \frac{\mathbf{p}^2}{m^2c^2} \right) = \mathbf{q} \cdot \mathbf{p} \frac{E^2}{m^2c^4} \tag{37}$$

Thus

$$q_i = x_i - \frac{p_i c^2}{E^2} \mathbf{x} \cdot \mathbf{p} \tag{38}$$

Similarly, the equations

$$[t, p_i] = \frac{\partial t}{\partial q_i} = \frac{E p_i}{m^2c^4} \tag{39}$$

with the boundary condition $t \to \theta$ as $v/c \to 0$ are solved by

$$t = \theta + \frac{E}{m^2c^4} \mathbf{q} \cdot \mathbf{p} \tag{40}$$

where θ is a time parameter satisfying

$$[\theta, p_i] = 0 \qquad \text{and} \qquad [\theta, q_i] \to 0 \quad \text{as} \quad v/c \to 0 \tag{41}$$

The last equation shows that θ can be chosen as a constant dynamical variable (independent of q and p), that is, a "c-number" satisfying

$$[\theta, q_i] = [\theta, p_i] = 0 \tag{42}$$

The transformation properties of θ under a Lorentz boost are particularly simple. Using equation (37) to replace $\mathbf{q} \cdot \mathbf{p}$ in equation (40) by $\mathbf{x} \cdot \mathbf{p}$, we find

$$\theta E = tE - \mathbf{x} \cdot \mathbf{p} = \theta' E' = \tau mc^2 \tag{43}$$

where the last equality is obtained by transforming to the particle's rest frame.

We now turn to the equations of motion. For a free particle, we have by equation (43),

$$d\theta = dt - \frac{d\mathbf{x} \cdot \mathbf{p}}{E} = dt\left(1 - \frac{v^2}{c^2}\right) \tag{44}$$

Hence, using

$$[x, E] = \frac{v_x}{1 - v^2/c^2} = \frac{dx}{dt} \frac{1}{1 - v^2c^2} \tag{45}$$

we find

$$\frac{dt}{d\theta} = [x, E] \tag{46}$$

Substitution of this result in equation (38) yields for a free particle

$$\frac{dq_i}{d\theta} = [q_i, E] \tag{47}$$

together with

$$\frac{dp_i}{d\theta} = [p_i, E] = 0 \tag{48}$$

Thus, for a general dynamical variable $A(q, p, \theta)$, we find

$$\frac{dA}{d\theta} = \frac{\partial A}{\partial \theta} + \sum \left(\frac{\partial A}{\partial q_i} \frac{\partial E}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial E}{\partial q_i} \right) = \frac{\partial A}{\partial \theta} + [A, E] \tag{49}$$

As mentioned in the introduction, in the quantum mechanical case one should replace expressions (36) and (40) by the symmetrized Hermitian operators

$$x_i = q_i + \frac{1}{2m^2c^2}\{p_i(\mathbf{q}\cdot\mathbf{p}) + (\mathbf{p}\cdot\mathbf{q})p_i\} \tag{50}$$

and

$$t = \theta + \frac{1}{2m^2c^4}\{E(\mathbf{q}\cdot\mathbf{p}) + (\mathbf{p}\cdot\mathbf{q})E\} \tag{51}$$

4. QUANTIZATION OF LENGTH

By construction, $\mathbf{q}$ and $\mathbf{x}$ behave as vectors under rotation. Indeed, it can be checked that the corresponding operators satisfy the expected commutation relations with the components of the angular momentum $\mathbf{L} = \mathbf{q} \times \mathbf{p}$. That is,

$$[L_i, x_j] = i\hbar\varepsilon_{ijk}x_k \tag{52}$$

In addition, it is not difficult to show that the noncommuting operators x_i satisfy

$$[x_i, x_j] = \frac{i\hbar}{m^2c^2}\varepsilon_{ijk}L_k \tag{53}$$

(Snyder, 1947).[3] Thus, the six Hermitian operators x_i, L_i form a closed algebra. By forming suitable linear combinations

$$\alpha x_i + \beta L_i = J_i \tag{54}$$

where J_i are supposed to be angular momentum operators, it is possible to decompose this algebra into a direct sum of two $O(3)$ algebras. Indeed, the equation

$$[\alpha x + \beta L_x, \alpha y + \beta L_y] = i\hbar[\alpha z + \beta L_z] \tag{55}$$

has two solutions $\alpha = \pm mc/2$ and $\beta = 1/2$ corresponding to

$$J_i^{(\pm)} = \pm\frac{mc}{2}x_i + \frac{1}{2}L_i \tag{56}$$

It is easy to check that $[J_i^{(+)}, J_j^{(-)}] = 0$.

[3]There is a marked similarity between some of the results in the present work and those of Snyder's. The two works differ, however, in their basic assumptions (e.g. three-dimensional versus four-dimensional dynamic spaces). I am grateful to Editor David Finkelstein for drawing my attention to the work of Snyder.

Since $[L_z, z] = 0$, the operators L_z and J_z (where $J_z \equiv J_z^{(+)}$) commute and thus can be diagonalized simultaneously. It therefore follows that the only possible outcomes for

$$z = \frac{1}{mc}(2J_z - L_z) \tag{57}$$

are

$$z = \frac{\hbar}{mc}(2\mu - M) \tag{58}$$

where 2μ and M are integers. Thus, for a particle of mass m, the length is an integral multiple of the Compton wavelength $\lambda_C = \hbar/mc$.

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